

### Ch.3. Existence and Uniqueness of solution:

§3) Lipschitz condition: (or Cauchy-Lipschitz condition).

A function  $f(x,y)$ , defined and continuous in a region  $D$  of the  $xy$ -plane, is said to satisfy Lipschitz condition in  $D$  if  $\exists$  a positive constant  $K$  such that  $|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$

whenever the points  $(x, y_1)$  and  $(x, y_2)$  both lie in  $D$ . The least value of the constant  $K$  is called the (a) Lipschitz constant for the function  $f(x,y)$  in  $D$ , and this is written as  $f \in \text{Lip}$ .

[ The condition  $\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} \leq K$   
 $\Rightarrow \frac{\partial f}{\partial y}$  is bounded — (i)  
Also  $f$  is continuous in  $D$  — (ii) ]

Note: If  $f(x,y)$  satisfies the condition

$$\left| \frac{\partial f}{\partial y} \right| \leq M \quad \text{--- (i)}$$

for all values of  $(x,y)$  in the given <sup>region</sup> range  $R$ , then for the same constant  $M$ , the Lipschitz condition is also satisfied.

Pf By M.V.T.M, we may write

$$f(x,y_2) - f(x,y_1) = (y_2 - y_1) \frac{\partial f}{\partial y} \Big|_{y=\eta} \quad \text{where } y_1 < \eta < y_2 \quad \text{--- (ii)}$$

and  $(x,y_1)$  and  $(x,y_2)$  are assumed to lie in the given <sup>region</sup> range  $R$ .

Now from (i) & (ii) we see

$$|f(x,y_2) - f(x,y_1)| \leq M|y_2 - y_1| \quad \text{--- (iii)}$$

which is Lipschitz condition.

It may be noted that Lipschitz condition (iv) may be replaced by a stronger condition (i). The converse of this result may not be true.

§3.2 Picard's Theorem: Existence and Uniqueness Thm.

Statement 1. Let  $f(x, y)$  be continuous in a domain  $D$  of the  $xy$ -plane and let  $M$  be a constant such that  $|f(x, y)| \leq M$  in  $D$  — (1)

Let  $f(x, y)$  satisfies in  $D$  the Lipschitz condition in  $y$ , namely

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|, \quad (2)$$

where the constant  $K$  is independent of  $x, y_1, y_2$ .

Let the rectangle  $R$ , defined by  $|x - x_0| \leq h, |y - y_0| \leq k$

lie in  $D$ , where  $Mh < k$ . Then for  $|x - x_0| \leq h$ , the differential eqn  $\frac{dy}{dx} = f(x, y)$  has a unique soln  $y = y(x)$  for which  $y(x_0) = y_0$ .

Statement 2. (As in G.I.U. class notes) Let  $f(x, y)$  be continuous and satisfies the Lipschitz condition  $|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$ . If  $(x_0, y_0)$  is any point of the strip  $a \leq x \leq b, -\infty < y < \infty$ , then the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \quad (1)$$

has one and only one solution on the interval  $a \leq x \leq b$ .

Ex.3.1. Show that the sol<sup>n</sup> of the initial value problem  
 $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$  may not be unique although  
 $f(x, y)$  is continuous. [G.U.]

Pf- we know that if  $f(x, y)$  is continuous and  
 satisfy the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2| \quad \text{--- (1)}$$

has one and only one sol<sup>n</sup>  $y = y(x)$  on the interval  
 $a \leq x \leq b$ ,  $-\infty < y < \infty$

Now, from (1) it is clear that as  $y_1 \rightarrow y_2$ ,  
 $f(x, y_1) \rightarrow f(x, y_2)$  in which case

$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| \text{ is undefined.}$$

which means  $\left| \frac{\partial f}{\partial y} \right|$  is undefined, and  
 we may not have unique sol<sup>n</sup>.

for example, we take

$$\frac{dy}{dx} = y^{1/3}, \quad y(0) = 0$$

it can readily be shown that the above  
 problem has two different sol<sup>n</sup>

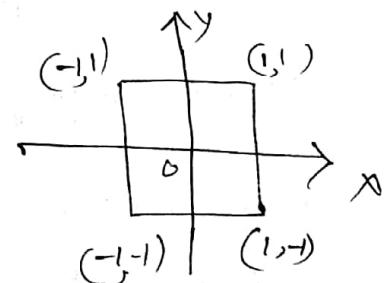
$$y_1(x) = \left(\frac{2}{3}x\right)^{3/2} \quad \& \quad y_2(x) = 0, \quad \text{in } |x| \leq 1, |y| \leq 1$$

though  $f(x, y) = y^{1/3}$  is continuous in  $|x| \leq 1, |y| \leq 1$

The non-uniqueness of sol<sup>n</sup>  
 lies on the fact that

$$\frac{\partial f}{\partial y} = \frac{1}{3y^{2/3}}$$

is undefined as  $y \rightarrow 0$



Thm. 3.3. Let the function  $f$  and  $\frac{\partial f}{\partial y}$  be continuous in some rectangle  $\alpha \leq x \leq \beta$ ,  $\gamma \leq y \leq \delta$  containing the point  $(x_0, y_0)$ . Then in some interval  $x_0 - h < x < x_0 + h$  contained in  $\alpha \leq x \leq \beta$ , there is a unique solution contained in  $\alpha \leq x \leq \beta$ , of the initial value problem  $y' = f(x, y)$ ,  $y = \phi(x)$  of the initial value problem  $y' = f(x, y)$ ,  $y(x_0) = y_0$ .

$$\text{Ex. 3.2} \quad \frac{dy}{dx} = x^2 + y^2, \quad y(1) = 3.$$

Here  $f(x, y) = x^2 + y^2$ ,  $\frac{\partial f}{\partial y} = 2y$  are continuous in every domain of the  $xy$ -plane.  
 The initial condition  $y(1) = 3$  means that  $x_0 = 1$ , and  $y_0 = 3$  and the point  $(1, 3)$  certainly lies in such domain. That is, there is a unique solution of the initial value problem in  $|x-1| \leq h$  about  $x=1$ .

$$\text{Ex. 3.3} \quad \frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0. \quad [\text{NET}]$$

Here  $f(x, y) = x^2 + y^2$  &  $\frac{\partial f}{\partial y} = 2y$  are continuous in the domain where  $(0, 0)$  is included.  
 we have unique soln in  $|x-0| \leq \epsilon$  i.e.  $|x| \leq \epsilon$

$$\text{Ex. 3.4} \quad \frac{dy}{dx} = \frac{y}{\sqrt{x}}, \quad y(1) = 2$$

Here  $f(x, y) = \frac{y}{\sqrt{x}} \rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x}}$  are both continuous in a domain containing the point  $(1, 2)$ . So there is unique soln in  $|x-1| \leq \epsilon$ .

Ex. 3.5.  $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$ ,  $y(0) = 2$ .

Further  
Here  $f(x,y) = \frac{y}{\sqrt{x}}$  now  $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x}}$  is continuous at  $x=0$   
 $y=2$ . Therefore the point  $(0,2)$  cannot be included  
in a domain to get a unique soln. Thus we conclude  
that the problem has a unique soln if  $(0,2)$  is not  
included in the domain.